CS 70   Discrete Mathematics and Probability Theory
Fall 2019   Alistair Sinclair and Yun S. Song   Discussion 8

1 The Count

How many of the first 100 positive integers are divisible by 2, 3, or 5?

2 Two Proof Methods

Consider the following identity:
$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

1. Prove the identity by algebraic manipulation (using the formula for the binomial coefficients).

2. Prove the identity using a combinatorial argument.

3 Monty Hall Challenge

Let us take on the challenge posed in lecture, and formally analyze the Monty Hall Problem.

(a) Assume that the corgi (the prize) and two goats were placed uniformly at random behind the three doors. What is the probability space $(\Omega, \mathbb{P})$?

(b) If our contestant chose door 1 in the first round, and decides to switch to another door after being shown a goat behind door 2 or 3, what are the events $C_1 =$"They win the corgi" and $\overline{C_1} =$"They win a goat"? What are their probabilities $\mathbb{P}(C_1)$ and $\mathbb{P}(\overline{C_1})$?

(c) If the contestant does not switch doors, what are the events $C_2, \overline{C_2}$ of winning the corgi and goats, and their respective probabilities now?

(d) If instead of choosing door 1 in the beginning, they chose a door uniformly at random, how do your $\Omega, \mathbb{P}, C_i, \overline{C_i}$ from above change?

4 Sampling

Suppose you have balls numbered $1, \ldots, n$, where n is a positive integer ≥ 2, inside a coffee mug. You pick a ball uniformly at random, look at the number on the ball, replace the ball back into the coffee mug, and pick another ball uniformly at random.

(a) What is the probability that the first ball is 1 and the second ball is 2?

(b) What is the probability that the second ball's number is strictly less than the first ball's number?

(c) What is the probability that the second ball's number is exactly one greater than the first ball's number?

(d) Now, assume that after you looked at the first ball, you did *not* replace the ball in the coffee mug (instead, you threw the ball away), and then you drew a second ball as before. Now, what are the answers to the previous parts?