Due: Tuesday, October 29, 2019 at 10:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

Note: *This homework consists of two parts. The first part (questions 1-6) will be graded and will determine your score for this homework. The second part (questions 7-8) will be graded if you submit them, but will not affect your homework score in any way. You are strongly advised to attempt all the questions in the first part. You should attempt the problems in the second part only if you are interested and have time to spare.*

For each problem, justify all your answers unless otherwise specified.

Part 1: Required Problems

1 Captain Combinatorial

Please provide combinatorial proofs for the following identities.

(a) $\binom{n}{i} = \binom{n}{n-i}$.

(b) $\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$.

(c) $\sum_{i=1}^{n} i\binom{n}{i}^2 = n\binom{2n-1}{n-1}$. (Hint: Part (a) might be useful.)

(d) $\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{n-i}\binom{n-i}{j} = 3^n$. (Hint: consider the number of ways of splitting n elements into 3 groups.)

2 On to Counting

Let us consider two finite sets A and B of cardinalities $|A| = n$ and $|B| = m$, respectively, and ask ourselves how many functions $f : A \to B$ there are that are surjective.

(a) Define F to be the set of all functions $f : A \to B$ (not necessarily surjective). What is the cardinality of F?

(b) For a fixed $b \in B$, define the set $F_b = \{f \in F \ : \ f^{-1}(\{b\}) = \varnothing\}$. What is the cardinality of F_b? How many functions in F_b are surjective? If f is not surjective, is it necessarily contained in $F_{b'}$ for some $b' \in B$?

(c) Use your results from the previous parts to compute the the number of functions from A to B that are surjective.

3 Flippin' Coins

Suppose we have an unbiased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 1/2$ and probability of tails also $\mathbb{P}[T] = 1/2$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?

(b) Which of the following are examples of *events*? Select all that apply.

- $\{(H,H,T),(H,H),(T)\}$
- $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
- $\{(T,T,T)\}$
- $\{(T,T,T),(H,H,H)\}$
- $\{(T,H,T),(H,H,T)\}$

(c) What is the complement of the event $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}$?

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

(e) What is the probability of the outcome (H,H,T)?

(f) What is the probability of the event that our outcome has exactly two heads?

4 Probability Warm-Up

(a) Suppose that we have a bucket of 30 red balls and 70 blue balls. If we pick 20 balls out of the bucket, what is the probability of getting exactly k red balls (assuming $0 \le k \le 20$) if the sampling is done with replacement?

(b) Same as part (a), but the sampling is without replacement.

(c) If we roll a regular, 6-sided die 5 times. What is the probability that at least one value is observed more than once?

5 Monty Halls

For each of the following modified Monty Hall scenarios, decide whether the contestant should switch doors or not. Unless otherwise specified, Monty, as in the original Monty Hall show, reveals a goat behind one door after the contestant has made their first choice.

(a) There are $n > 2$ doors with 1 car and $n-1$ goats.

(b) There are $n > 2$ doors with 1 car and $n-1$ goats, but Monty reveals $n-2$ doors with goats behind them.

(c) There are $n > 2$ doors with $k < n-1$ cars and $n-k$ goats, but Monty reveals $j < n-k$ doors with goats behind them. For what k and j is the relative advantage of switching doors largest? That is, for what values of k and j is the ratio of winning when switching doors to winning when not switching doors largest?

6 Past Probabilified

For the following experiments, please (i) define an appropriate sample space Ω, (ii) give the probability function $\mathbb{P}$, and (iii) compute $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$ for the two given events E_1 and E_2.

(a) Fix a prime $p > 2$, and sample twice with replacement from $\{0, \ldots, p-1\}$, then multiply these two numbers with each other in $(\bmod p)$ space. $E_1 =$ The resulting product is 0, $E_2 =$ The product is $(p-1)/2$.

(b) Sample a random graph on n vertices by including every possible edge with probability $1/2$. $E_1 =$ The graph is complete, $E_2 =$ vertex v_1 has degree d.

(c) Create a random stable marriage instance by having each person's preference list be a random permutation of the opposite gender. Finally, create a random pairing by matching men and women up randomly. $E_1 =$ The resulting pairing is the female-optimal stable pairing, $E_2 =$ All men have distinct favorite women.

Note: *This concludes the first part of the homework. The problems below are optional, will not affect your score, and should be attempted only if you have time to spare.*

Part 2: Optional Problems

7 Fermat's Wristband

Let p be a prime number and let k be a positive integer. We have beads of k different colors, where any two beads of the same color are indistinguishable.

(a) We place p beads onto a string. How many different ways are there construct such a sequence of p beads with up to k different colors?

(b) How many sequences of p beads on the string are there that use at least two colors?

(c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p.)

[*Hint*: Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

(d) Use your answer to part (c) to prove Fermat's little theorem.

8 Peaceful rooks

A friend of yours, Eithen Quinn, is fascinated by the following problem: placing m rooks on an $n \times n$ chessboard, so that they are in peaceful harmony (i.e. no two threaten each other). Each rook is a chess piece, and two rooks threaten each other if and only if they are in the same row or column. You remind your friend that this is so simple that a baby can accomplish the task. You forget however that babies cannot understand instructions, so when you give the m rooks to your baby niece, she simply puts them on random places on the chessboard. She however, never puts two rooks at the same place on the board.

(a) Assuming your niece picks the places uniformly at random, what is the chance that she places the $(i+1)^{\text{st}}$ rook such that it doesn't threaten any of the first i rooks, given that the first i rooks don't threaten each other?

(b) What is the chance that your niece actually accomplishes the task and does not prove you wrong?

(c) If you were using checker pieces as a replacement for rooks (so that they can be stacked on top of each other), then what would be the probability that your niece's placements result in peace? Assume that two pieces stacked on top of each other threaten each other.

(d) Explain the relationship between your answer to the previous part and the birthday paradox. In particular if we assume that 23 people have a 50% chance of having a repeated birthday (in a 365-day calendar), what is the probability that your niece places 23 stackable pieces in a peaceful position on a 365×365 board?