CS 70 Discrete Mathematics and Probability Theory
Spring 2016 Walrand and Rao Final

PRINT Your Name: ________________________________, ________________________________

(last) (first)

SIGN Your Name: ________________________________

PRINT Your Student ID: ________________________________

WRITE your exam room:

Name of the person sitting to your left: ________________________________

Name of the person sitting to your right: ________________________________

- After the exam starts, please write your student ID (or name) on every odd page (we will remove the staple when scanning your exam).
- We will not grade anything outside of the space provided for a problem unless we are clearly told in the space provided for the question to look elsewhere. Scratch paper is not allowed to be used for answers.
- Give justifications as noted.
- You may consult three single-sided sheet of notes. Apart from that, you may not look at books, notes, etc. Calculators, phones, and computers are not permitted.
- There are 17 single sided pages on the exam. Notify a proctor immediately if a page is missing.
- **You may, without proof, use theorems and facts that were proven in the notes and/or in lecture.**
- **You have 180 minutes: there are 36 problems on this exam.**
  - **Problems 1-3: 20 problems. 80 points.**
  - **Problem 4-6: 16 problems. 40 points.**

Do chill, but also focus and keep moving through the exam as well.

Do not turn this page until your instructor tells you to do so.

1. Short Questions: 3/3/5/5 Provide a clear and concise justification of your answer.

In this problem, you flip a coin that is such that $Pr[H] = p$. For questions 1-2-3(a), express the answers in terms of p. For questions 3(b) and 4, we are looking for answers such as $2\%, 4\%$, or the like.

1. If you flip the coin three times, what is the probability that the number X of heads exceeds the number of tails? (Express in terms of p.)

2. How many times do you have to flip the coin, on average, until you see the fifth 'heads'. (The fifth head counts as a flip, of course. Express in terms of p.)

3. Assume $p < 0.5$. (a) Use Chebyshev's inequality to get an upper bound on the probability that when you flip the coin 100 times the number of heads exceeds the number of tails. (Again, express in terms of p.) (b) Find the value when $p = 0.4$. (Evaluate to a numeric value.)

4. Assume $p = 0.4$. Use the CLT to get an estimate of the probability that when you flip the coin 100 times the number of heads exceeds the number of tails. (Use the fact that $\sqrt{0.4 \times 0.6} \approx 0.5$ and evaluate to a numeric value.)

2. Short Questions: 3/3/3/3/3 Provide a clear and concise justification of your answer.

When asked for justification for true/false. If true, prove or justify. If false, give a counterexample.

1. True or False (and justification): Let X, Y, Z be three random variables defined on the same probability space. If $cov(X,Y) \geq 0$ and $cov(Y,Z) \geq 0$, then it must be that $cov(X,Z) \geq 0$.

2. True or False (and justification): Say that you roll a six sided die 100 times and the total number of dots you get is 350. Given that information, the expected number of dots on the next roll is 3.5.

3. True or False (and justification): Let X, Y be two random variables such that $E[Y|X] = 2 + 3X$. Then $L[Y|X] = 2 + 3X$.

4. True or False (and justification): Let X, Y be two random variables such that $E[Y|X] = 2 + 3X$. Then it must be that $E[Y^2|X] = 4 + 12X + 9X^2$.

5. True or False (and justification): Consider an irreducible Markov chain $\{X_n, n \geq 0\}$ on $\{1, 2, \ldots, 17\}$ with a uniform invariant distribution and assume that $P(1,1) = 0.2$. Is it true that $P[X_n \leq 5]$ as $n \to \infty$ converges to $\frac{5}{17}$.

3. Longer Questions: 8/8/8/7/6/6/6 Provide a clear and concise justification of your answer.

1. (Short Answers - No justification required.) The left-hand side of Figure 2 shows the six equally likely values of the random pair (X,Y). For questions $A-D$, specify the functions.

   [A]. $L[Y|X] =$
   [B]. $E[X|Y] =$
   [C]. $L[X|Y] =$
   [D]. $E[Y|X] =$

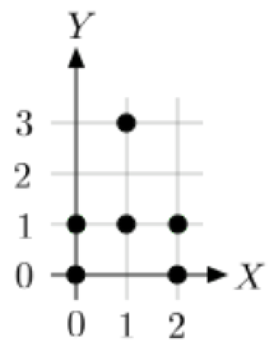

Figure 1: (X,Y) in Question 3.1

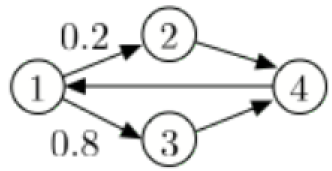

Figure 2: Markov Chain in Question 3.2

2. (Multiple Choice - No justification - Indicate which statements are correct by checking the corresponding boxes). The right-hand part of Figure 2 shows the state transition diagram of a Markov chain.

   [ ] A. This Markov chain is irreducible and aperiodic.

   [ ] B. This Markov chain is irreducible and periodic.

   [ ] C. $\pi = [0.25, 0.25, 0.25, 0.25]$ is invariant for the Markov chain.

   [ ] D. $\pi = [1/3, 1/15, 4/15, 1/3]$ is invariant for the Markov chain.

   [ ] E. If $\pi_0 = [1, 0, 0, 0]$, then π_n converges as $n \to \infty$.

   [ ] F. If $\pi_0 = [1, 0, 0, 0]$, then π_n does not converge as $n \to \infty$.

   [ ] G. If $\pi_0 = [1, 0, 0, 0]$, then $(1/n)\sum_{m=0}^{n-1} 1\{X_m = 1\}$ does not converge as $n \to \infty$.

   [ ] H. If $\pi_0 = [1, 0, 0, 0]$, then $(1/n)\sum_{m=0}^{n-1} 1\{X_m = 1\}$ converges as $n \to \infty$.

3. You flip a fair coin. What is the average number of tails (not flips!) until you get (tails, heads, heads) in a row (i.e., THH)?

   Note: You will get 75% of the credit if you write correctly the relevant first step equations and an additional 25% if you can solve them.

4. Consider a population of N healthy people. Every week, a healthy person gets sick with probability $\alpha \in (0,1)$ and a sick person gets healthy with probability $\beta \in (0,1)$. For $n \geq 1$, let X_n be the number of sick people at the start of week n.

   (a) Find $E[X_{n+1}|X_n]$.

   (b) Find $E[X_n]$ for $n \geq 1$.

   (c) As $n \to \infty$, what does $Pr[X_n = m]$ converge to?

5. You play a game of darts with a friend. You are better than he is and the distances of your darts to the center of the target are i.i.d. $U[0,1]$ whereas his are i.i.d. $U[0,2]$. To make the game fair, you agree that you will throw one dart and he will throw two darts. The dart closest to the center wins the game. What is the probability that you will win? *Note:* The distances ***from the center of the board*** are uniform..

6. There are two indistinguishable bags. Bag A contains 60 red marbles and 40 blue marbles. Bag B contains 40 red marbles and 60 blue marbles. One chooses one of the two bags at random, so that each bag has probability 0.5 of being selected. One then removes 3 marbles from the selected bag, without replacement.

   (a) Assume that all the three marbles were red. What is the probability that the selected bag is A? *Note: We don't want you to evaluate numerically the expression.*

   (b) Assume that r of the 3 marbles were red and b were blue and let p be the conditional probability that the selected bag is A given that information. What is the probability that if you pick one more ball from the same bag it will be red? *Note: The expression will contain* (p, r, b) *and we do not want you to express* p *as a function of* r *and* b.

7. We throw n balls in m bins where $n > 0$ and $m \geq 2$. Each ball is equally likely to drop into any one of the bins, independently of other balls. Let X and Y be the number of balls that fall into bins 1 and 2, respectively.

   (a) Calculate $E[Y|X]$.

   (b) Calculate $L[Y|X]$.

   (c) Write the first step equations to calculate the expected number of balls one has to throw until one bin contains 2 balls. (*Hint:* What is the Markov chain that you consider? Do not solve the equations.)

4. True/False (All 1 point)

1. $(Q \implies P) \equiv (\neg(Q \wedge \neg P))$

2. $(((\forall x)P(x)) \implies ((\forall y)Q(y))) \equiv (((\exists y)\neg Q(y)) \implies (\neg(\forall x)P(x)))$

3. $(R \wedge \neg R) \implies P$

4. The problem of computing whether P and Q have the same behavior (that is, outputs the same thing as $P(x)$ for all x and loops on x when $P(x)$ does not halt), is decidable/undecidable. (Please answer either decidable or undecidable.)

5. The problem of computing whether P and Q have the same behavior on all inputs of length at most $|P|$ is decidable/undecidable? (Please answer either decidable or undecidable.)

6. The problem of computing whether P and Q have the same behavior on all inputs of length $|P|$ and where the space used by $|P|$ is at most $|P|^2$ is decidable/undecidable? (Please answer either decidable or undecidable.)

5. What number (or expression)? 1/1/2/2/2/2/2/2/2/2)

1. The number of binary strings of length n.

2. The number of binary strings of length n (where n is even) with a run of at least $n/2$ zeros.

3. The number of ways to split n dollars among k people where at most one gets zero dollars.

4. An n packet message was sent through a channel by sending m points on a degree $n-1$ polynomial. If k packets were lost, what is the maximum number of the remaining $m-k$ packets that could be corrupted so that the original message can still be recovered.

5. How many degree d polynomials, $P(x)$, modulo p are there with exactly d solutions to $P(x) = 5 \pmod{p}$? (You may assume that p is prime and at least 5 and that $d < p$.)

6. What is $2^{53} \pmod{15}$?

7. What is the maximum degree of an undirected graph whose vertices are $\{0,\ldots,p-1\}$ edges $\{(i,i+1 \pmod{p}) : i \in \{0,\ldots,p-1\}\}$? (You can assume p is prime.)

8. What is the maximum degree of an undirected graph, $G(a,p)$, whose vertices are $\{0,\ldots,p^2-1\}$, and edges $\{(i,ai \pmod{p^2}) : i \in \{0,\ldots,p^2-1\}\}$? Assume $a \neq 0 \pmod{p^2}$. (Give an expression that possibly depends on a and p and properties thereof. Again, p is prime, but a may not be, so your formula may have cases.)

9. Given a connected planar graph with f faces, if one adds an edge and it remains planar, what is the number of faces in the resulting planar graph? (An expression perhaps involving f.)

10. Let S be the set of all planar bipartite graphs with v vertices. What is the maximum number of edges a graph in S can have? Express it in terms of v.

6. Proofs and Algorithm (4 points each.)

1. Prove that for all primes p that if $p \not| x^2$ then $p \not| x$.

2. Consider two degree d polynomials $P(x)$ and $Q(y)$ modulo p, and the two-variable polynomial $f(x,y) = P(x)Q(y)$. (***You may assume that you can efficiently recover a degree*** d ***polynomial from*** $d+1$ ***points or anything else we proved about polynomials in this class.***)

   Say you are given the value of $f(x,y)$ on $(1,1),\ldots,(2d+1,2d+1)$ and on $(1,1),\ldots,(1,d+1)$, give a method to reconstruct $Q(y)$ and $P(x)$. You may assume that the leading coefficients of $Q(y)$ and $P(x)$ are 1.

3. Given a set of k triples from a set of vertices $\{1,\ldots,n\}$ (e.g., $(2,4,10)$ is a triple when $n \geq 10$) where each $i \in \{1,\ldots,n\}$ is in at most d triples, describe an algorithm that uses at most $3d-2$ colors to legally color the triples so that no two triples that contain the same vertex have a common color.

   (Note: two $(2,4,6)$ and $(2,8,10)$ both contain vertex 2 and thus cannot have the same color. We are looking for a two line algorithm including justification. A line or two more is ok.)

4. Prove that a number is divisible by 3 if and only if the sum of its digits is divisible by 3. (Hint: One way to prove this is to strengthen an inductive statement into a statement about the value of a number modulo 3.)