CS 70 Discrete Mathematics and Probability Theory
Fall 2012 Vazirani **Midterm 2 Solutions**

1 Counting

For each question write the answer as an expression involving factorials, powers and binomial coefficients, and a one sentence explanation:

1. The number of ways of choosing 6 cards from a standard deck of cards to get three pairs.

–Solution–

Choose 3 distinct ranks for the 3 pairs, then for each rank, choose 2 out of 4 suits to make up the pair:

$$\binom{13}{3} \cdot \binom{4}{2}^3$$

2. The number of bridge hands with a $(6,3,2,2)$ distribution. i.e. 6 cards in one suite, 3 in another, and 2 each in the remaining suites.

–Solution–

Choose the 6-card suit from the 4 available suits, then choose the 3-card suit from the remaining 3 suits. The remaining 2 suits must be the (indistinguishable) 2-card suits. Then choose the appropriate number of cards from each suit:

$$4 \cdot 3 \cdot \binom{13}{6} \cdot \binom{13}{3} \cdot \binom{13}{2} \cdot \binom{13}{2}$$

3. The number of anagrams of CANDIDATE.

–Solution–

First label each letter to be distinct, e.g., $\text{CA}_0\text{ND}_0\text{ID}_1\text{A}_1\text{TE}$. There are 9! permutations of the 9 distinct labeled letters. Now, for each distinct permutation of the original word CANDIDATE, there are exactly $2! \cdot 2!$ permutations of the labeled word, corresponding to the different ways to rearrange the two A's and the two D's in place. Thus, the number of permutations of the original word is:

$$\frac{9!}{2!2!}$$

4. The number of rolls of 6 dice with exactly 4 distinct values.

–Solution–

First, choose the four values (out of the six sides of the die) that will appear. Then, multiply by the number of ways to put 6 indistinguishable balls (the rolls) in 4 distinguishable bins (the values), with at least one ball in each bin.

$$\binom{6}{4} \cdot \binom{6-1}{4-1} = \binom{6}{4}\binom{5}{3}$$

5. The number of (strictly) increasing sequences of k numbers from $\{1, 2, \ldots, n\}$. e.g. the sequence $1, 3, 4, 7, 20$ is (strictly) increasing.

–Solution–

For every choice of k numbers from $\{1, 2, \ldots n\}$, there is exactly one way to draw them in strictly increasing order. So just:

$$\binom{n}{k}$$

6. The number of integers $x:\ 0 \leq x < 41$ such that $4x = 21(\text{mod} 41)$.

–Solution–

4 is relatively prime to 41, so it has an additive inverse mod 41. Multiply both sides of this equation by the additive inverse to solve for the unique value of x. Therefore:

$$1$$

7. The number of seven digit numbers where the first digit is not 0 and no pair of adjacent digits are the same.

–Solution–

There are 9 choices for the first digit. For every digit after the first, there are 9 choices (anything but the previous digit). Therefore:

$$9^7$$

2 Calculating Probabilities

The Tigers are down 1-2 in the World series (i.e. the Tigers have won 1 and the Giants 2 of the first 3 games out of a best of seven games series). The Tigers are still think they stand a good chance of winning the series since they estimate that their chances of winning any given game is 0.6 (the outcomes of different games are independent).

1. If you believe their assumption, what is the probability that the Tigers will win the series?

–Solution–

> There are two ways for the Tigers to win. They can win the next 3 games, with probability $.6^3$. Alternately, they can win 3 times and lose once, e.g., win-win-lose-win. Each such sequence of wins and losses has probability $.6^3 \cdot .4$, and there are 3 such sequences (the loss can be any game except the seventh and last, since in that case, they have already won the series). Thus, the probability that they win is $.6^3 + 3 \cdot .6^3 \cdot .4 \approx .475$.

2. The assistant coach of the Tigers believes that the situation is quite dire, and the Tigers should pull out all stops in Game 4. He reckons that if they put in a super human effort they can improve their chances of winning to with 0.75. The coach points out that if they do so, they will be so mentally spent that their chances of winning in subsequent games will drop to 0.5. What is their chance of winning the tournament under this strategy?

   Remember you may leave your answers as expressions involving factorials, etc.

–Solution–

> Let W denote the event that the Tigers win the series, and F the event that they win the fourth game. By the Total Probability Rule:
>
> $$P(W) = P(W|F)P(F) + P(W|\neg F)P(\neg F)$$
>
> If F occurs, the Tigers need to win two out of the three remaining games to take the Series. This can take the form win-win (probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$), win-lose-win (probability $(\frac{1}{2})^3 = \frac{1}{8}$, or lose-win-win (probability $\frac{1}{8}$ again). Thus, $P(W|F) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$. If $\neg F$ occurs, they need to win all 3 remaining games, with probability $\frac{1}{8}$. We can now compute:
>
> $$P(W) = \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{8} \cdot \frac{1}{4} = \frac{13}{32} \approx .406$$

3 Lines

For this question, assuming that you are working modulo p, where p is a prime greater than 10. Select a random line (a polynomial $A(x)$ of degree at most 1).

1. What is the chance that it goes through a particular point, (x, y), for example if $(x, y) = (0, 5)$, the question asks what is the probability that $A(0) = 5$?

–Solution–

The total number of distinct lines (polynomials of the form $ax + b$) is p^2, since we can independently choose a and b. (Some of these lines will be constant functions, or "flat" lines).

How many of these lines go through (x, y)? By Lagrange interpolation, every distinct value of y' in $\{0, 1, \ldots p-1\}$, there is a distinct line connecting (x, y) and $(x+1, y')$; moreover, every line passing through (x, y) must be one of those p lines. Thus, there are exactly p such lines, and the probability is $\frac{p}{p^2} = \frac{1}{p}$.

2. What is the chance that it goes through two particular points $(x_1, y_1),\ (x_2, y_2)$, where $x_1 \neq x_2$?

–Solution–

Again by Lagrange interpolation, of the p^2 lines, there is exactly 1 connecting those 2 points. Thus the probability is $\frac{1}{p^2}$.

3. What about 3 particular points $(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3)$, where $x_1,\ x_2,\ x_3$ are distinct?

–Solution–

There are two distinct cases. Use Lagrange interpolation to obtain the line connecting (x_1, y_1) and (x_2, y_2). If (x_3, y_3) lies on this line (i.e., the 3 points are collinear), there is exactly 1 such line, so the probability is $\frac{1}{p^2}$. If it does not, there is no such line and the probability is 0.

4 Graphs:

A *c*omplete bipartite graph is an undirected graph whose vertex set is the union of two disjoint non-empty sets, V_0 and V_1, and whose edges consist of all pairs $\{u, v\}$: $u \in V_0$ and $v \in V_1$. (there are no edges between pairs of vertices in V_0 or pairs of vertices in V_1)

1. Prove that a complete bipartite graph is connected. i.e. there is a path between every pair of vertices.

–Solution–

Consider two arbitrary vertices v, v'; we wish to show that there is a path between them. If $v \in V_0$ and $v' \in V_1$ (or vice versa), there is an edge connecting them, so (v, v') is already a path.

If v and v' are both in V_0, then choose any $u \in V_1$ (by the assumption that V_1 was nonempty); then (v, u, v') is a path from v to v'. (The case where they are both in V_1 is analogous.)

2. Prove that for any graph, G, either G or $\bar{G}$ (the complement of G) is connected. Here $\bar{G}$ has the same vertex set as G, but a pair of vertices is adjacent in $\bar{G}$ iff there is no edge between them in G.

–Solution–

Consider an arbitrary graph G. If G is connected, we are done. So assume G is disconnected. We need to show that $\bar{G}$ is connected.

Let v, v' be arbitrary vertices in G; we wish to show that there is a path between them in $\bar{G}$. If they are in different connected components of G, then they are connected by an edge in $\bar{G}$. If they are in the same connected component of G, choose any u in a distinct connected component of G; such a u must exist by the assumption that G is disconnected. Then, $\bar{G}$ has an edge (v, u) and an edge (u, v'), so (v, u, v') is a path between v and v' in $\bar{G}$.

5 Democratic convention

At the democratic convention, President Obama is scheduled to give his acceptance speech at an outdoor stadium. In recent years, on average it has rained on only 10% of the evenings in September. Unfortunately, the weatherman has predicted rain for the evening of the speech. Going through the historical records, on evenings when it rained, the weatherman's predictions were correct 80% of the time. And on evenings when it did not rain he incorrectly forecast rain 20% of the time. What is the probability that it will rain on the evening of President Obama's acceptance speech? Write your answer as a rational number, i.e. in the form $\frac{a}{b}$.

–Solution–

Let R be the event that it rains at the convention, and W the event that the weatherman predicts rain. Translating the information we have been given into the language of probability, we have:

- $P(R) = \frac{1}{10}$
- $P(W|R) = \frac{4}{5}$ ("when it rained, the prediction was correct 80% of the time")
- $P(W|\neg R) = \frac{1}{5}$ ("when it did not rain, he incorrectly forecast rain 20% of the time")

We want to compute $P(R|W)$. Using Bayes' rule:

$$P(R|W) = \frac{P(W|R)P(R)}{P(W)}$$

We do not yet know $P(W)$, but we can compute it using the Total Probability Rule:

$$P(W) = P(W|R)P(R) + P(W|\neg R)P(\neg R) = \frac{4}{5} \cdot \frac{1}{10} + \frac{1}{5} \cdot \frac{9}{10} = \frac{13}{50}$$

thus

$$P(R|W) = \frac{\frac{4}{5} \cdot \frac{1}{10}}{\frac{13}{50}} = \frac{4}{13} \approx .31$$

Intuitively, even though the weatherman predicted rain, the background probability of rain is low enough that his prediction is most likely a false positive.