Midterm 2

6:00-8:00pm, 15 April

Notes: There are **six** *questions on this midterm. Answer each question part in the space below it, using the back of the sheet to continue your answer if necessary. If you need more space, use the blank sheet at the end. In both cases, be sure to clearly label your answers!* **None of the questions requires a long answer, so avoid writing too much! Unclear or long-winded solutions may be penalized.** *The approximate credit for each question part is shown in the margin (total 60 points). Points are not necessarily an indication of difficulty!*

Your Name: **Your Section:**

Person on Your Left: **Person on Your Right:**

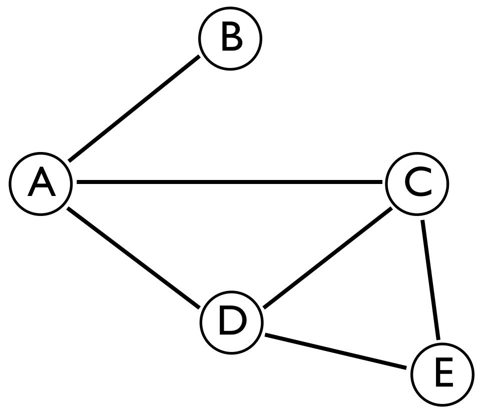
For official use; please do not write below this line!

Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
Total	

[exam starts on next page]

1. **Euclid's tunnels** [7 points]

The inhabitants of a medieval French village, plagued by marauding bandits, constructed a system of underground tunnels which allowed them to travel safely between key points in the village. The network of tunnels is shown in the following figure, where A is Anne's House, B is Bluebeard's Castle, C is Conomor's Bar, D is the Dancehall, and E is Euclid's Residence:



Answer the following questions. **In each case you should give a brief explanation, with reference to results from class—not an ad hoc argument.**

(a) Anne frequently wants to travel from her house to the dancehall. On each trip, in preparation for her exertions on the dance floor, she wants to take some exercise by walking along all the tunnels (without traversing any tunnel twice), ending up at the dance hall. To be able to do this, Anne realizes that she must request that a new tunnel be added to the network. Between which two points should this tunnel be built? *3pts*

(b) Bluebeard also wants to make regular trips from his castle to the dancehall, and he also wants to traverse every tunnel once for the quite different purpose of catching unsuspecting villagers. After Anne's tunnel is built, Bluebeard realizes that, by destroying one tunnel, he can achieve his aim. Which tunnel should he destroy? *2pts*

(c) After Anne's tunnel has been built as in part (a), and Bluebeard's has been destroyed as in part (b), the village council decides, in the interests of fairness to all, that the network should be reconfigured so that it is possible for anybody, starting at any point, to traverse all the tunnels once and end up back at their starting point. An astute councillor notes that this can be achieved by constructing one additional tunnel. Where should this tunnel be built? *2pts*

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2. Counting, counting... [12 points]

For all parts of this question, simplify your answer as much as possible and **circle the final answer**. If the final answer is an arithmetic expression involving binomial coefficients or factorials, **do not** simplify it further. For each part, provide a **brief** explanation of your answer.

(a) Find the number of six letter words that can be formed using letters of the word ELEPHANT. [HINT: Consider separately the case when both E's are included.] *3pts*

(b) Given a natural number n, find the number of triples (a, b, c) where a, b, c are natural numbers such that $0 \leq a \leq b \leq c \leq n$. *3pts*

(c) Find the number of ways to create three committees out of a group of n citizens such that all citizens serve on at least one committee and no citizen serves on all three committees. (Some committees may be empty.) *3pts*

(d) Out of n students taking an exam, two students are known to be very good friends. Find the number of ways of seating the n students in a line so that the two friends are not seated next to each other. *3pts*

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3. A random number of dice throws [12 points]

A fair six-sided die is tossed N times, where N is chosen randomly so that $\Pr[N = 1] = \frac{1}{2}$, $\Pr[N = 2] = \frac{1}{4}$, $\Pr[N = 3] = \frac{1}{8}$, and $\Pr[N = 4] = \frac{1}{8}$. Let X be the sum of the scores obtained from all N tosses. Answer the following questions.

Note: Do not attempt to simplify your answers; they should be left as sums, products and quotients of fractions—as, for example, in the expression $\frac{\frac{1}{2}\cdot\frac{3}{4}}{\frac{1}{4}\cdot\frac{1}{3}+\frac{2}{5}\cdot\frac{4}{5}}$. Be sure to circle your final answer, and to explain your calculations.

(a) What is the probability that $X = 3$ given that $N = 2$? *3pts*

(b) What is the probability that $X = 3$? *3pts*

(c) What is the probability that $X = 3$ given that N is even? *3pts*

(d) What is the probability that $N = 2$ given that $X = 3$? *3pts*

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4. Locks and keys [10 points]

There are n boxes each having a lock and a key. A prankster randomly places one key inside each box and then locks up all the boxes. After the prankster departs, we open box 1 with a duplicate key and recover the key contained in it. The recovered key is used to open another box, and we continue until we find a box with the key to box 1 inside it, at which point we stop. Answer the following questions, **showing your working in each case.**

(a) What is the probability that we open exactly 1 box? *2pts*

(b) What is the probability that we open exactly 2 boxes? *2pts*

(c) What is the probability that we open exactly k boxes? *3pts*

(d) What is the expected number of boxes opened? *3pts*

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5. Random graphs [9 points]

A *random undirected graph* G with n vertices $\{1, 2, \ldots, n\}$ is formed by including each possible edge $\{u, v\}$ with probability p, independently of all other edges. Answer the following questions, giving each answer in terms of n and p. Use tidy factorials or binomial coefficients where necessary.

(a) Recall that a *Hamiltonian cycle* is a cycle that goes through every vertex exactly once. What is the expected number of Hamiltonian cycles in G? *3pts*

(b) Recall that a *Hamiltonian path* is a path that visits every vertex *exactly once*. What is the expected number of Hamiltonian paths that start at vertex 1 and end at vertex n? *3pts*

(c) Fix an integer $k < n$. What is the expected number of simple paths of length exactly k (i.e., having k edges)? (Note that different paths of length k can have some edges in common.) *3pts*

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6. A Declaration of (In)dependence [10 points]

You are given N coins, where N is selected from a Poisson distribution with parameter λ. All the coins have probability p of landing Heads. Let X and Y be the numbers of Heads and Tails respectively when all the N coins are tossed once.

(a) What is the distribution of X given that $N = 100$? *2pts*

(b) Using the law of total probability, write an expression for $\Pr[X = k]$, in terms of k and λ. (Your solution should involve a summation—DO NOT simplify your solution.) *2pts*

(c) Since a Poisson process is giving rise to the coin flips, we know that for each tiny time interval there is some (tiny) chance of flipping a Heads, and hence we would expect the distribution of X also to be Poisson. [This can be proved by simplifying the expression in part (b), but you are NOT required to do this.] Assuming that the distribution of X is Poisson, write a simple expression for $\Pr[X = k]$. [HINT: What should the parameter of X's Poisson distribution be?] *2pts*

(d) Compute $\Pr[X = k | Y = i]$, using the definition of conditional probability and parts (a) and (c). Your solution should not contain a summation. **Show your work.** *3pts*

(e) For natural numbers j, k, are the events $X = k$ and $Y = j$ independent? Explain your answer in one sentence. *1pt*

[The End!]