Midterm 2

7:00-9:00pm, 12 April

Your Name: **Your Section:**

Person on Your Left: **Person on Your Right:**

Instructions:

(a) *There are* **five** *questions on this midterm.*

(b) **Question 1** *consists of several parts, each requiring true-false or multiple choice answers. Indicate your answers in the space provided for each part; you do* **not** *need to show your working for Question 1.*

(c) *For* **Questions 2–5***, you should write your answer to each part in the space below it, using the back of the sheet to continue your answer if necessary. If you need more space, use the blank sheet at the end. In both cases, be sure to clearly label your answers!*

(d) **None of the questions requires a long answer, so avoid writing too much! Unclear or long-winded solutions may be penalized.**

(e) *The approximate credit for each question part is shown in the margin (total 70 points). Points are not necessarily an indication of difficulty!*

For official use; please do not write below this line!

Q1	
Q2	
Q3	
Q4	
Q5	
Total	

[exam starts on next page]

1. True-False/Multiple Choice [26 points]

Each of the following questions requires either a True-False or Multiple Choice answer. For the True-False questions, write your answer in the **box** *provided. For the Multiple Choice questions, circle the* **one** *correct answer. You do* **not** *need to show your working. For the* **True-False questions** *incorrect answers will receive negative scores, so if you are unsure of your answer leave it blank;* **do not guess!**

(a) **[True or false?]** Mark each of the following statements "True" or "False" in the box provided: *11pts*

[ ] In an undirected graph, the sum of the degrees of the vertices is equal to the number of edges in the graph.

[ ] In a directed graph, the sum of the in-degrees of the vertices is equal to the sum of the out-degrees.

[ ] The number of edges in an n-dimensional hypercube is $n2^{n-1}$.

[ ] The length of a de Bruijn sequence is always a power of two.

[ ] The following is a de Bruijn sequence for $n = 3$: 10010110.

[ ] In any probability space, there is an event E such that E and A are independent for all non-empty events A.

[ ] In any probability space, there is an event E such that E and A are *not* independent for all non-empty events A.

[ ] For any events A and B such that $\Pr[B] > 0$, we have $\Pr[A|B] \geq \Pr[A \cap B]$.

[ ] Let A and B be events such that $\Pr[A|B] = \Pr[A|\overline{B}]$, where $\overline{B}$ denotes the complement of B. Then A and B are independent.

[ ] For events A, B, the probability that neither of the events happens is $1 - \Pr[A] - \Pr[B]$.

[ ] For three events A, B, C, the probability that *exactly one* of the events happens is $\Pr[A] + \Pr[B] + \Pr[C] - \Pr[A \cap B] - \Pr[B \cap C] - \Pr[A \cap C] + 2\Pr[A \cap B \cap C]$.

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(b) **[True or false?]** Let X and Y be random variables on the same probability space with expectations $\mathrm{E}(X) = 2$ and $\mathrm{E}(Y) = 1$. Which of the following statements must be true? *4pts*

☐ $\mathrm{E}\big((2X+1)(Y+1)\big) = 2\mathrm{E}(XY) + 6$

☐ $\mathrm{E}(1/X) = 1/2$

☐ $\mathrm{E}(X^2) = 4$

☐ $\Pr[X > Y] > 0$

(c) **[Multiple choice]** You are dealt a hand of five cards from a randomly shuffled standard deck of 52 cards. *2pts*

(i) The probability that your hand contains no pair of cards with the same numerical value (there are 13 numerical values: ace, 2, . . ., king) is

$\left(\frac{12}{13}\right)^4$ $\qquad \frac{48\cdot 47\cdot 46\cdot 45}{51\cdot 50\cdot 49\cdot 48}$ $\qquad \frac{48\cdot 44\cdot 40\cdot 36}{51\cdot 50\cdot 49\cdot 48}$ $\qquad \frac{13\cdot 12\cdot 11\cdot 10\cdot 9}{52\cdot 51\cdot 50\cdot 49\cdot 48}$ $\qquad 4\cdot\frac{12\cdot 11\cdot 10\cdot 9}{51\cdot 50\cdot 49\cdot 48}$

(ii) The probability that the hand contains *at least three* cards of the same value is

$\frac{13\left(4\cdot\binom{48}{2}+48\right)}{\binom{52}{5}}$ $\qquad \frac{13\cdot 4\cdot 48\cdot 47}{\left.{52 \atop 5}\right)}$ $\qquad \frac{13\cdot 48}{\binom{52}{5}}$ $\qquad \left(\frac{1}{13}\right)^3$ $\qquad \frac{13\cdot\binom{5}{3}\cdot\binom{4}{3}}{\binom{52}{5}}$

(d) **[Multiple choice]** A fair six-sided die is thrown repeatedly. *3pts*

(i) The expected number of throws until the first 6 appears is

$1 \qquad 3 \qquad 7/2 \qquad 6 \qquad 12$

(ii) The expected number of throws until two 6's have appeared is

$2 \qquad 3 \qquad 6 \qquad 7 \qquad 12$

(iii) The expected number of throws until *two different* numbers have appeared is

$2 \qquad 11/5 \qquad 3 \qquad 7 \qquad 12$

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(e) **[Multiple choice]** A book contains 10^6 characters. Each character is mis-typed with probability 10^{-5}, independently of all other characters. *2pts*

(i) The expected number of mis-typed characters in the book is

$$10^{-6} \qquad 10^{-5} \qquad 1 \qquad 10 \qquad 100$$

(ii) The probability that the book contains exactly seven mis-typed characters is approximately

$$\frac{7}{10} \qquad \frac{7!}{10^7} \qquad 7 \times 10^{-5} \qquad 7 \times e^{-10} \qquad e^{-10}\frac{10^7}{7!}$$

(f) **[Multiple choice]** The n volumes of an encyclopedia are arranged in a random sequence on a shelf, so that all $n!$ orderings are equally likely. Let $\pi(i)$ denote the position of volume i in the sequence. *2pts*

(i) The probability that $\pi(1) < \pi(2) < \pi(3)$ is

$$\frac{1}{\binom{n}{3}} \qquad \frac{1}{n} \qquad \frac{1}{6} \qquad \frac{1}{3} \qquad \frac{1}{2}$$

(ii) *Conditional on* the event $\pi(1) < \pi(3)$, the probability that $\pi(1) < \pi(2) < \pi(3)$ is

$$\frac{1}{\binom{n}{2}} \qquad \frac{1}{n} \qquad \frac{1}{6} \qquad \frac{1}{3} \qquad \frac{1}{2}$$

(g) **[Multiple choice]** Each cereal box contains one coupon, chosen independently and uniformly at random from a set of n different coupons. *2pts*

(i) The expected number of boxes that need to be bought before at least one copy of *all* n coupons is obtained is on the order of

$$n^2 \qquad n \ln n \qquad \sqrt{n} \qquad n \qquad \mathrm{e}^n$$

(ii) The number of boxes that need to be bought before the probability that some coupon appears twice reaches 1/2 is on the order of

$$n^2 \qquad n \ln n \qquad \sqrt{n} \qquad n \qquad \mathrm{e}^n$$

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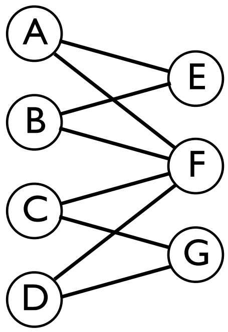
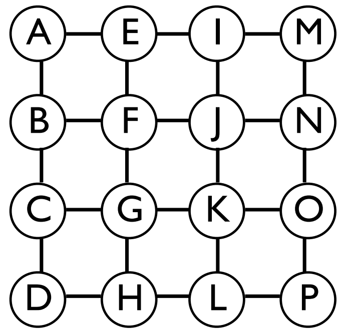
2. Euler and Hamilton [8 points]

This question concerns the following two undirected graphs. For each graph, say whether it has an **Eulerian cycle** and also whether it has a **Hamiltonian cycle**. In each case, if the respective cycle exists indicate the cycle by listing the vertices visited in order, starting from vertex A; if the cycle does not exist, give a **short** explanation for why it does not exist. You may use without proof any results from class provided they are clearly stated.



(a)



(b)

(a) Answer for graph in figure (a): *4pts*

(b) Answer for graph in figure (b): *4pts*

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3. Artistic Counting [14 points]

Write your answers as succinctly as you can, but **do not** attempt to evaluate them. For example, you should leave your answers in terms of factorials and binomial coefficients. **Write your final answer for each part in a box! You do not need to explain your answers.**

(a) A painter allows each of his 10 most loyal patrons to choose one free painting from his catalog of 50 original paintings. (There is only one version of each original.) In how many ways can the patrons choose? *2pts*

(b) In the above scenario, what is the number of possible sets of 40 remaining (unchosen) paintings? *2pts*

(c) The painter is particularly fond of two of his paintings, and he is not willing to give both of them away (but giving either one away is OK). How does this change the answer to part (a) above? *2pts*

(d) The painter now reconsiders and decides to offer reproductions rather than the original paintings. Each painting can be reproduced arbitrarily many times, but each patron may still only choose one painting. In how many ways can the patrons now choose (ignoring the constraint in part (c))? *2pts*

(e) Once the patrons have selected their reproductions, the painter compiles a request for the printer specifying how many copies of each painting are needed. How many different possible requests are there? *2pts*

(f) Upon further reconsideration, the painter decides to offer each patron five reproductions rather than just one (and these five choices need not be distinct). How does this change the answer to part (d) above? *2pts*

(g) As a final compromise, the painter instead decides that each patron may pick one original and one reproduction, and that these two choices must be distinct. In how many ways may the patrons now choose? *2pts*

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4. One Double-Headed Coin [12 points]

A woman has three coins, one of which has Heads on both sides and the other two of which each have one Head and one Tail. All coins are unbiased. Answer the following questions; **explain your reasoning briefly and write your final answers in a box!**

(a) The woman closes her eyes, picks one coin at random, and tosses it. What is the probability that it comes up Heads? *3pts*

(b) The woman opens her eyes and sees that the coin has come up Heads. What is the probability that the other (hidden) side of this coin is Heads? *3pts*

(c) The woman now picks up the coin and tosses it again. What is the probability that this second toss also comes up Heads? *3pts*

(d) After observing only the two Heads tosses of the first coin, the woman discards this coin, closes her eyes, picks one of the remaining two coins at random and tosses it. What is the probability that it comes up Heads? *3pts*

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5. Light Bulb Factory [12 points]

A light bulb factory has three machines, A, B, C, all of which produce identical light bulbs. Machine A accounts for 50% of the production, machine B for 30%, and machine C for 20%. Moreover, 4% of the bulbs produced by machine A, 2% of those produced by machine B, and 5% of those produced by machine C are defective. Answer the following questions; **explain your reasoning clearly and write your final answers in a box!**

(a) The factory owner gives you a light bulb chosen at random from the factory's output. What is the probability that the bulb is defective? *4pts*

(b) The random bulb you are given turns out to be defective. What are the probabilities that the bulb was produced by machine A, machine B, and machine C respectively? *4pts*

(c) The foreman gives you a second bulb produced by the same machine as the first one. What is the probability that this second bulb is also defective? *4pts*

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6. Random Networking [10 points]

A set of $n \geq 3$ people $\{1, 2, \ldots, n\}$ attend a networking event. Upon arrival, each person is given a hat whose color is equally likely to be one of $m \geq 2$ colors, independent of all the other people. The idea is that, to get the event started, every pair of people with hats of the same color will introduce themselves.

(a) For each pair of people i, j with $i < j$, let the indicator random variable $X_{\{i,j\}}$ be 1 if i and j receive hats of the same color, and 0 otherwise. Are the random variables $X_{\{i,j\}}$ mutually independent? Briefly justify your answer. *3pts*

(b) Let the random variable X be the number of pairs of people who have hats of the same color. What is the expectation $\mathrm{E}(X)$, as a function of n and m? *3pts*

(c) Focus now on one person, Jason Nerd. Let the random variable Y be the number of people with the same color hat as Jason (not including Jason himself). What is the distribution of Y? *2pts*

(d) What is the expectation $\mathrm{E}(Y)$? *2pts*

[The End!]