1 Expectation and Variance

This problem will give you some practice calculating expectations and variances of random variables. Suppose that the random variable X takes on 3 values, $10, 25, 70$. Suppose $\mathbb{P}[X = 10] = 0.5$, $\mathbb{P}[X = 25] = 0.2$, and $\mathbb{P}[X = 70] = 0.3$.

(a) What is $\mathbb{E}[X]$?

(b) What is $\mathbb{E}[X^2]$?

(c) What is $\mathrm{Var}(X)$?

2 Runs

Suppose I have a biased coin which comes up heads with probability p, and I flip it n times, recording the sequence of heads and tails that I see. A "run" in this sequence is a subsequence of coin flips all of the same type, which is not contained in any longer sequence of coin flips all of the same type. For example, the sequence "HHHTHH" has three runs: "HHH," "T," and "HH." Compute the expected number of runs in a sequence of n flips.

3 Airport Revisited

(a) Suppose that there are n airports arranged in a circle. A plane departs from each airport, with probability $1/2$ flies to the airport directly to its left, and with probability $1/2$ to the one directly to its right. What is the expected number of empty airports after all planes have landed?

(b) Now suppose that we still have n airports, but instead of being arranged in a circle, they form graph, where each airport is denoted by a vertex, and an edge between two airports indicates that a flight is permitted between them. There is a plane departing from each airport and randomly chooses a neighboring destination where a flight is permitted. What is the expected number of empty airports after all planes have landed? (Express your answer in terms of $N(i)$, the set of neighboring airports of airport i, and $\deg(i)$, the number of neighboring airports of airport i).

4 Joint Distributions

(a) Give an example of discrete random variables X and Y with the property that $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$. You should specify the joint distribution of X and Y.

(b) Give an example of discrete random variables X and Y that (i) are *not independent* and (ii) have the property that $\mathbb{E}[XY] = 0$, $\mathbb{E}[X] = 0$, and $\mathbb{E}[Y] = 0$. Again you should specify the joint distribution of X and Y.