1   Allen's Umbrella Setup

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

(a) Model this as a Markov chain. What is $\mathscr{X}$? Write down the transition matrix.

(b) What is the transition matrix after 2 trips? n trips? Determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

2   Can it be a Markov Chain?

(a) A fly flies in a straight line in unit-length increments. Each second it moves to the left with probability 0.3, right with probability 0.3, and stays put with probability 0.4. There are two spiders at positions 1 and m and if the fly lands in either of those positions it is captured. Given that hte fly starts between positions 1 and m, model this process as a Markov Chain.

(b) Take the same scenario as in the previous part with $m = 4$. Let $Y_n = 0$ if at time n the fly is in position 1 or 2 and let $Y_n = 1$ if at time n the fly is in position 3 or 4. Is the process Y_n a Markov chain?

3 Three Tails

You flip a fair coin until you see three tails in a row. What is the average number of heads that you'll see until getting TTT?

4 Markov Property Practice

Let $X_0, X_1, \ldots$ be a Markov chain with state space S, such that i_j is the value that X_j takes in the jth state. One of the properties that it satisfies is the Markov property:

$$\mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = \mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1}), \text{ for all } i_0, i_1, \ldots, i_n \in S, \ n \in \mathbb{Z}_{>0}.$$

Use the Markov property and the total probability theorem to prove the following.

(a) $\mathbb{P}(X_3 = i_3 | X_2 = i_2, X_1 = i_1) = \mathbb{P}(X_3 = i_3 | X_2 = i_2)$, for all $i_1, i_2, i_3 \in S$.

   Note: This is not exactly the Markov property because it does not condition on X_0.

(b) $\mathbb{P}(X_3 = i_3 | X_1 = i_1, X_0 = i_0) = \mathbb{P}(X_3 = i_3 | X_1 = i_1)$, for all $i_0, i_1, i_3 \in S$.

(c) $\mathbb{P}(X_1 = i_1 | X_2 = i_2, X_3 = i_3) = \mathbb{P}(X_1 = i_1 | X_2 = i_2)$, for all $i_1, i_2, i_3 \in S$.