CS 70 Discrete Mathematics and Probability Theory
Fall 2019 Alistair Sinclair and Yun S. Song **DIS 6**

1 Count It!

For each of the following collections, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

(a) $\mathbb{Z}$, the set of all integers.

(b) $\mathbb{Q}$, the set of all rational numbers.

(c) The integers which divide 8.

(d) The integers which 8 divides.

(e) The functions from $\mathbb{N}$ to $\mathbb{N}$.

(f) Numbers that are the roots of nonzero polynomials with integer coefficients.

2 Countability Basics

(a) Is $f : \mathbb{N} \to \mathbb{N}$, defined by $f(n) = n^2$, an injection (one-to-one)? Briefly justify.

(b) Is $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^3 + 1$, a surjection (onto)? Briefly justify.

3 Hilbert's Paradox of the Grand Hotel

Consider a magical hotel with a countably infinite number of rooms numbered according to the natural numbers where all the rooms are currently occupied. Assume guests don't mind being moved out of their current room as long as they can get to their new room in a finite amount of time. In other words, guests can't be moved into a room that's infinitely far from the current one.

(a) Suppose one new guest arrived in their car, how would you shuffle guests around to accommodate them? What if k guests arrived, where k is a constant positive integer?

(b) Suppose a countably infinite number of guests arrived in an infinite length bus with seat numbers according to the natural numbers, how would you accommodate them?

(c) Suppose a countably infinite number of infinite length buses arrive, each carrying a countably infinite number of guests. How would you accommodate them?

4 Interval Bijection Construction

Recall that:

- (a,b) denotes the open interval of real numbers r such that $a < r < b$,
- $[a,b)$ denotes the half-closed interval of real numbers r such that $a \leq r < b$,
- and $[a,b]$ denotes the closed interval of real numbers r such that $a \leq r \leq b$.

(a) Construct a bijective function $f : [0,1] \to (0,1]$.

(b) Construct a bijective function $g : [0,1] \to (0,1)$.

5 True or False?

For all $n \in \mathbb{N}$, let $\mathscr{F}_n$ denote the set of all funtions from $\{0,1,\ldots,n\}$ to $\{0,1\}$. Let $\mathscr{F}$ denote the set of all functions from $\mathbb{N}$ to $\{0,1\}$. Then is the following statement true or false:

$$|\mathscr{F}| = |\cup_{n \in \mathbb{N}} \mathscr{F}_n|?$$

Justify your answer.