CS 70 Discrete Mathematics and Probability Theory
Fall 2013 Midterm #1

PRINT your name: ______________________________, ______________________________
(last) (first)

SIGN your name: ____

PRINT your student ID: ____

Name of the person sitting to your left: ___________________________

Name of the person sitting to your right: ___________________________

You may consult one single-sided sheet of paper with notes. Apart from that, you may not look at books, notes, etc. Calculators and computers are not permitted.

Please write your answers in the spaces provided in the test. We will not grade anything on the back of an exam page or outside the space provided for a question unless we are clearly told on the front of the page in the space provided for the question to look there.

Please write your name and student ID on every page.

You have 80 minutes. There are 4 questions, of varying credit (75 points total), as well as one extra-credit question. Use the number of points as a rough guide for the amount of time to allocate to that question (5 minutes for checking your answers). Good luck!

Do not turn this page until your instructor tells you to do so.

1. **(30 points) Multiple choice/Numerical Answer. No justification necessary**

   (a) Suppose you proved the inductive step for a statement $P(n)$ but then discovered that $P(29)$ is false. What can you say about $P(1)$?
   Necessarily true **Necessarily false** **Cannot say**

   (b) Suppose you proved the inductive step for a statement $P(n)$ but then discovered that $P(29)$ is false. What can you say about $P(50)$?
   Necessarily true **Necessarily false** **Cannot say**

   (c) The polynomial $x^2 - 1$ mod 15 has at most 2 zeros.
   True **False**

   (d) The polynomial $x^2 - 1 \mod 31$ has at most 2 zeros.
   True **False**

   (e) $gcd(a,b) = gcd(a,b+25a)$.
   True **False**

   (f) $gcd(a,b) = gcd(2a,b+2a)$.
   True **False**

   (g) What is the multiplicative inverse of 7 mod 13?

   (h) 15 has a multiplicative inverse $\mod 78$.
   True **False**

   (i) What is $5^{547} \mod 15$?

   (j) Circle all that apply. The function $f(x) = x^3$ mod 21 is:
   One-to-one **Onto** **Bijection** **None of the previous**

   (k) The inverse of the function $f(x) = 5x$ mod 21 is $g(x) = 17x$ mod 21.
   True **False**

2. **(15 points) Stable Marriage.**

   Suppose that after running a stable marriage algorithm with n men and n women, the pairing that results includes the couple (1, A). Suppose that after a few days 1 changes his mind, and decides that he does not like woman A as much as he thought he did (i.e. he moves her down on his preference list). What is the maximum number of rogue couples that result in the existing pairing from such a change to 1's preference list? Give a one or two sentence justification for why the number of rogue couples can be as large as you claim. Also give a one or two sentence justification for why the remaining couples cannot be rogue couples.

3. **(15 points) Well-Ordering Principle.**

   Suppose I start with a necklace with three beads: one red, one green, and one blue. Each day I cut the necklace at an arbitrary point and then lay it out in a line. I look at the beads on the two ends of the necklace. If the end beads are the same color, I throw away my necklace. If the end beads are different colors, then I add a new bead of the third color to one end (for example, if one bead was red and one bead was green, I add a blue bead) and retie the necklace.

   Use the well-ordering principle to prove that I never have to throw away my necklace (i.e. prove that regardless of my choices, there is no day where I end up throwing my necklace).

4. **(15 points) Lagrange Interpolation.** You are doing Lagrange Interpolation to find a polynomial $P(x)$ of degree 10, using the data $(1,P(1)),(2,P(2)),\ldots,(11,P(11))$. You discover that $P(x)=\sum_{i=1}^{11}\Delta_i(x)$. What is $P(20)$? Justify your answer. Ideally your justification of your answer will be at most 3-4 sentences.

   Hint: Recall that $\Delta_i(x)$ is a polynomial of degree 10 such that $\Delta_i(i)=1$ and $\Delta_i(j)=0$ for all other j in $\{1,\ldots,11\}$. If you are having trouble getting started, you might try plotting one of the polynomials $\Delta_i(x)$, say $\Delta_4(x)$, at these values of x.

5. **(1 point) Bonus Problem.** Prove that $\sqrt{2\sqrt{3\sqrt{4\cdots\sqrt{n}}}} < 3$.

   (*Hint:* what can you say about the quantity $\sqrt{k\sqrt{(k+1)\cdots\sqrt{n}}}$?).