Midterm 1

8:00-10:00pm, 24 September

Your First Name: **Your Last Name:**

SIGN Your Name: **Your SID Number:**

Your Exam Room:

Name of Person Sitting on Your Left:

Name of Person Sitting on Your Right:

Name of Person Sitting in Front of You:

Name of Person Sitting Behind You:

Instructions:

(a) *As soon as the exam starts, please write your student ID in the space provided at the top of every page! (We will remove the staple when scanning your exam.)*

(b) *There are 6* **double-sided** *sheets (12 numbered pages) on the exam. Notify a proctor immediately if a sheet is missing.*

(c) *We will not grade anything outside of the space provided for a question (i.e., either a designated box if it is provided, or otherwise the white space immediately below the question).* **Be sure to write your full answer in the box or space provided!** *Scratch paper is provided on request; however, please bear in mind that nothing you write on scratch paper will be graded!*

(d) *The questions vary in difficulty, so if you get stuck on any question it may help to leave it and return to it later.*

(e) *On questions 1-2: You need only give the answer in the format requested (e.g., True/False, an expression, a statement.) An expression may simply be a number or an expression with a relevant variable in it. For short answer questions, correct, clearly identified answers will receive full credit with no justification. Incorrect answers may receive partial credit.*

(f) *On questions 3-6, you should give arguments, proofs or clear descriptions if requested. If there is a box you must use it for your answer: answers written outside the box may not be graded!*

(g) *You may consult one two-sided "cheat sheet" of notes. Apart from that, you may not look at any other materials. Calculators, phones, computers, and other electronic devices are NOT permitted.*

(h) *You may, without proof, use theorems and lemmas that were proven in the notes and/or in lecture.*

(i) *You have 120 minutes: there are 6 questions on this exam worth a total of 135 points.*

[exam starts on next page]

1. **True/False** [*No justification; answer by shading the correct bubble. 2 points per answer; total of 32 points. No penalty for incorrect answers.*]

   (a) Let $P(x), Q(x)$ be propositions involving a natural number x. Suppose you are asked to prove that the following statement is FALSE: $(\forall x \in \mathbb{N})(P(x) \Rightarrow Q(x))$. Which of the following would constitute a valid proof strategy? Answer **YES** or **NO** for each by shading the appropriate bubble.

YES NO

○ ○ Find some x for which $P(x)$ holds and $Q(x)$ does not hold. *2pts*

○ ○ Find some x for which both $P(x)$ and $Q(x)$ hold. *2pts*

○ ○ Show that $P(x)$ holds for all x. *2pts*

○ ○ Show that $Q(x)$ doesn't hold for any x, and find some x for which $P(x)$ holds. *2pts*

○ ○ Show that $Q(x)$ holds for all x, and find some x for which $P(x)$ doesn't hold. *2pts*

   (b) The following True/False questions concern the stable marriage problem. The term "traditional SMA" denotes the propose-and-reject algorithm described in class, in which men propose to women. Answer **TRUE** or **FALSE** for each by shading the appropriate bubble.

TRUE FALSE

○ ○ A non-stable pairing can have more than one rogue couple. *2pts*

○ ○ For every problem instance, there exist at least two distinct stable pairings. *2pts*

○ ○ In the man-optimal stable pairing, it is possible for a man to be paired with his least favorite woman. *2pts*

○ ○ In the man-optimal stable pairing, it is possible for two men to be paired with their least favorite women. *2pts*

○ ○ If man M and woman W are each other's respective least favorite choices, then M cannot be matched with W in any stable pairing. *2pts*

○ ○ If the traditional SMA terminates after exactly k days on a given instance, then the women-optimal SMA (in which the women propose to the men) also terminates after exactly k days on the same instance. *2pts*

[Q1 continued on next page]

(c) Answer each of the following questions **TRUE** or **FALSE** by shading the appropriate bubble.

TRUE FALSE

◯ ◯ Every hypercube is bipartite. *2pts*

◯ ◯ A 10-dimensional hypercube has a cycle of length 5. *2pts*

◯ ◯ The complete bipartite graph $K_{2,n}$ is planar for every n. [Recall that, for integers $m, n \geq 1$, $K_{m,n}$ is the bipartite graph with m vertices in one part, n in the other part, and all possible edges between the two parts.] *2pts*

◯ ◯ For any integer a, the equation $3x \equiv a \pmod{101}$ always has a solution for x. *2pts*

◯ ◯ For any integer a, the equation $3x \equiv a \pmod{102}$ always has a solution for x. *2pts*

2. Short Answers [*Answer is a single number or expression; write it in the box provided; no justification necessary. 3 points per answer; total of 42 points. No penalty for incorrect answers.*]

(a) Suppose that $P(x, y)$ is a proposition that involves real numbers x and y. Write down quantified propositions that express each of the following statements:

(i) For some x, $P(x, y)$ holds for all y. *3pts*

(ii) For every x, there is exactly one y such that $P(x, y)$ holds. [Note: The only quantifiers you may use are $\forall$ and $\exists$.] *3pts*

(iii) For all x and y, if $P(x, y)$ holds then $P(y, x)$ does not hold. *3pts*

(b) What is the minimum possible number of edges in a *connected* graph on n vertices (with no self-loops or multiple edges between any pair of vertices)? *3pts*

(c) What is the maximum possible number of edges in a *disconnected* graph on n vertices (with no self-loops or multiple edges between any pair of vertices)? *3pts*

(d) A connected planar graph G with 6 vertices and 11 edges is drawn in the plane. How many faces does it have? *3pts*

(e) A connected planar graph G has 100 vertices. What is the maximum possible number of edges in G? *3pts*

[Q2 continued on next page]

(f) Find the rightmost (least significant) digit of 7^{93}. *3pts*

(g) Solve the equation $3x \equiv 7 \pmod 8$ for x. *3pts*

(h) Suppose the 25th day of this year is a Monday. Which day of the week is the 50th day of next year (assuming that this year has 365 days)? *3pts*

(i) What is $14^{201} \pmod{15}$? *3pts*

(j) Find $1! + 2! + 3! + 4! + \ldots + 100! \pmod{24}$. *3pts*

(k) Find $\gcd(481, 91)$. *3pts*

(l) Find the inverse of 27 mod 32. Your answer should be an integer in $\{0, 1, \ldots, 31\}$. *3pts*

3. Proofs [*All parts to be justified. Total of 18 points.*]

(a) Prove that $\sqrt{6}$ is irrational. *6pts*

(b) Prove by induction that, for all odd $n \geq 1$, $2^n + 1$ is divisible by 3. *6pts*

[Q3 continued on next page]

(c) The so-called "Tribonacci" sequence $T(n)$ is defined as follows: *6pts*

$$T(0) = T(1) = 0; \quad T(2) = 1; \quad T(n) = T(n-1) + T(n-2) + T(n-3) \text{ for } n \geq 3.$$

Prove by induction that $T(n) \leq 2^n$ for all $n \geq 0$.

4. Stable Marriage [*All parts to be justified unless otherwise stated. Total of 13 points.*]

Consider the following set of marriage preferences for four men 1, 2, 3, 4 and four women A, B, C, D.

Man	Women			
1	B	D	A	C
2	C	A	B	D
3	B	D	C	A
4	C	B	A	D

Woman	Men			
A	3	4	1	2
B	2	4	3	1
C	1	2	3	4
D	2	3	4	1

(a) Use the Propose-and-Reject algorithm to find a male-optimal pairing. Show your work in the table below: you should indicate in each column the proposals received by each woman on each day. Note: The algorithm may terminate in fewer than the 6 days provided! *4pts*

Woman	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6
A						
B						
C						
D						

(b) Is it possible for man 4 to be paired with woman C in a stable pairing? Briefly justify your answer using the optimality/pessimality properties of stable pairings. *3pts*

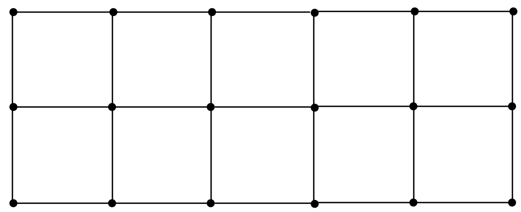
(c) Is it possible for man 2 to be paired with woman A in a stable pairing? Briefly justify your answer using the optimality/pessimality properties of stable pairings. *3pts*

[Q4 continued on next page]

(d) Suppose we add k new men and k new women to the above instance. The preference lists of these new people are completely arbitrary orderings on the $k + 4$ people of the opposite gender. The preference list of each of the original four men and four women starts off exactly as above, followed by an arbitrary ordering of the new people of the opposite gender. Prove that any stable pairing of the larger instance must include a stable pairing of the original 4-man, 4-woman instance above. [Note: This proof is fairly short; you probably won't need all the space below.] *3pts*

5. Grid Graphs. [*All parts to be justified unless otherwise stated. Total of 18 points.*]

For integers $m, n \geq 2$, an $m \times n$ *grid graph* is an undirected graph drawn in the plane, with mn vertices positioned at the integer points (i, j) for $1 \leq i \leq n$ and $1 \leq j \leq m$. An edge is drawn between each pair of points whose distance in the plane is exactly 1. The figure below shows a 3×6 grid graph.



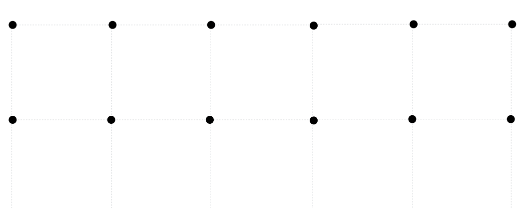
(a) How many *edges* are there in an $m \times n$ grid graph? Write your answer in the box provided; no justification is needed. *2pts*

(b) How many *faces* are there in an $m \times n$ grid graph? Write your answer in the box provided; no justification is needed. *2pts*

(c) Verify that the grid graph satisfies Euler's formula. *2pts*

(d) For which values of n and m does the grid graph have an Eulerian tour? Write your answer in the box provided; no justification is needed. *2pts*

(e) Recall that a *Hamiltonian cycle* in a graph is a (simple) cycle that visits every vertex exactly once. Draw a Hamiltonian cycle in the 3×6 grid graph below. [Note: You may use the diagram at the top of the page to experiment; please clearly show your final answer on the diagram below!] *2pts*



[Q5 continued on next page]

(f) Prove by induction on n that, for every *even* $n \geq 2$, every $m \times n$ grid graph has a Hamiltonian cycle. [Hint: Consider a systematic way of drawing a Hamiltonian cycle; strengthen your induction hypothesis.] *5pts*

(g) Prove that, if m and n are both odd, the $m \times n$ grid graph has *no* Hamiltonian cycle. [Hint: Give a direct proof, not a proof by induction.] *3pts*

6. Modular Arithmetic and Random Number Generation [*All parts to be justified unless otherwise stated. Total of 12 pts.*]

An important application of modular arithmetic is to generate a sequence of pseudo-random numbers $x_0, x_1, x_2, \ldots$, defined by the recursion

$$x_n = ax_{n-1} \pmod p, \text{ for } n = 1, 2, \ldots$$

Here p is a prime number, a is a positive integer such that $a \not\equiv 0 \pmod p$, and $x_0 \in \mathbb{Z}^+$ is a seed (initialization) satisfying $x_0 \not\equiv 0 \pmod p$. The *period* d is the smallest $n \in \mathbb{Z}^+$ such that $x_n \equiv x_0 \pmod p$; note that the sequence repeats after d numbers have been generated. We want to make d as large as possible: in this problem, you will see how large d can possibly be, using basic facts about modular arithmetic.

(a) For $n \in \mathbb{N}$, find x_n as a function of n, a, and x_0. Write your answer in the box provided; no justification is needed. *2pts*

(b) Prove that $a^d \equiv 1 \pmod p$. *2pts*

(c) Now let n_0 be the smallest positive integer n such that $a^n \equiv 1 \pmod p$. Prove that n_0 divides all positive integers n such that $a^n \equiv 1 \pmod p$. *4pts*

(d) Finally, prove that the period d divides $p - 1$. State clearly which results you used to prove this claim; in addition to earlier parts of this problem, you may use any of the theorems/lemmas from lecture. *4pts*