CS 70 Discrete Mathematics and Probability Theory
Fall 2011 Rao Midterm 2 Solutions

1 True/False. [24 pts]

Circle one of the provided answers please!
No negative points will be assigned for incorrect answers.

(a) TRUE or FALSE: Given independent events A, B where A and B have nonzero probability, then $A \cap B$ is nonempty.

True. Since the $Pr[A] > 0$, then the $Pr[A|B] > 0$. We need to ensure that $Pr[B] > 0$, since $Pr[A|B]$ is undefined when $Pr[B] = 0$.

(b) TRUE or FALSE: If A, B, and C are mutually independent, then $Pr[A|B,C] = Pr[A]$.

True. This follows from the definition of mutually independent as stated in note 11 in the reader.

(c) TRUE or FALSE: If $Pr[A|B] = 2Pr[A]$, then $Pr[B] > Pr[A]$.

False. Consider a probability space a uniform probability for outcome in $\Omega = \{1,2,3,4,5,6\}$ (a die roll) $B = \{1,2\}$ and $A = \{1,2,3\}$. Here, $Pr[A|B] = 1$ where $Pr[A] = \frac{1}{2}$ and $Pr[B] < Pr[A]$.

(d) TRUE or FALSE: It is necessarily true that the variance of a random variable X is $\leq (E(X))^2$.

False. Consider a random variable X with $Pr[X = 0] = .99$ and $Pr[X = 100] = .01$, $E(X) = 1$, $Var(X) = .99(1) + .01(10000) = 100.99$ which is greater than $(E(X))^2 = 1$.

(e) TRUE or FALSE: It is necessarily true that the variance of a random variable X is $\leq E(X^2)$.

True. A way to compute variance is $E(X^2) - (E(X))^2$. Since $(E(X))^2$ is nonnegative, the difference can be at most $E(X^2)$.

(f) TRUE or FALSE: For disjoint events A and B, the $Pr[A \cap B] = Pr[A] \times Pr[B]$.

False. For disjoint events the $Pr[A \cap B] = 0$ regardless of the $Pr[A]$ and $Pr[B]$. Any pair of disjoint events where each have nonzero probability provides an counterexample.

(g) TRUE or FALSE: For independent events, $Pr[A \cup B] = Pr[A] + Pr[B]$.

False. $Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A]Pr[B]$ from inclusion/exclusion, the intersection rule, and for independent A and B. Any two non-empty independent events provides a counterexample.

(h) TRUE or FALSE: For a Poisson random variable X with parameter λ, the $Pr[X = i+1] \leq Pr[X = i]$ for all $i \geq \lambda$.

True. For Poission random variables

$$Pr[X = i] = e^{-\lambda}\frac{\lambda^i}{i!}.$$

and

$$Pr[X = i+1] = e^{-\lambda}\frac{\lambda^{i+1}}{(i+1)!}.$$

The ratio is $\frac{\lambda}{i+1}$ which is smaller than 1 when $i \geq \lambda$.

(i) TRUE or FALSE: For a Poisson random variable X with parameter $\lambda = 1$, then Chebyshev's inequality ensures that the $Pr[X \geq 11] \leq \frac{1}{100}$.

True. We can see that if $X \geq 11$ *implies* $|X-1| \geq 10$. Thus $Pr[X \geq 11] \leq Pr[|X-1| \geq 10]$. By Chebyshev's inequality, we have that

$$Pr[|X-1] \geq 10] \leq \frac{Var(X)}{10^2} = \frac{1}{100}.$$

The last equality follows since variance of this Poisson distribution is $\lambda = 1$.

(j) TRUE or FALSE: For a binomially distributed variable X with parameter $p = \frac{1}{2}$ and $n = 100$, Chebyshev's inequality ensures that the $Pr[X \geq 75] \leq \frac{1}{10}$.

True. As above, $Pr[X \geq 75] \leq Pr[|X-50| \geq 75$ Since the mean of this binomial distribution is $pn = 50$ and the variance is $(1-p)pn = 25$, use Chebyshev to conclude

$$Pr[|X-50] \geq 25] \leq \frac{25}{25^2} = \frac{1}{25} \leq \frac{1}{10}$$

(k) TRUE or FALSE: Given two random variables, X with Poisson distribution and Y with a geometric distribution, both with mean μ, we can conclude that $E[X+Y] > E[2X]$.

False. By linearity of expectation $E[X+Y] = E[X]+E[Y] = 2\mu$ which is not greater than 2μ.

(l) TRUE or FALSE: The maximum variance binomial distribution with parameter n has parameter $p = 1$.

False. The variance for a binomial is $p(1-p)n = 0$ in this case. Any nonzero parameter for $p = 0$.

(m) TRUE or FALSE: Given a random variable $S = X_1 + \ldots + X_n$ where the X_i's are chosen independently from the same distribution, and any α, $Pr[|S - E[S]| \geq \alpha]$ goes to 0 as n goes to infinity.

False. The standard deviation here is $\sqrt{n}\sigma$ which goes to infinity as n goes to infinity. Thus, the distance from S to $E(S)$ does not "go to zero."

Some students noticed that α was not required to be positive in which case the statement is trivially false.

2 Short answer. [43 pts]

For parts a to b, consider a deck with just the four aces (red: hearts, diamonds; black: spades, clubs). Melissa shuffles the deck and draws the top two cards.

(a) [4 pts] Given that Melissa has the ace of hearts, what is the probability that Melissa has both red cards?

$\frac{1}{3}$. Each of the 6 hands is equally likely. 3 of them contain the ace of hearts. One of them contains two red cards. Thus, $Pr[\text{"two red"}|\text{"ace of hearts"}] = \frac{1}{3}$.

(b) [4 pts] Given that Melissa has at least one red card, what is the probability that she has both red cards?

$\frac{1}{5}$. There are 5 hands with at least one red card (since only one of the six hands has two black cards) . One of them contains two red cards. Thus, $Pr[\text{"two red"}|\text{"one red"}] = \frac{1}{5}$.

(c) [4 pts] Suppose that A and B are independent, C is disjoint from both A and B and $P[A] = P[B] = P[C] = 1/4$. Compute $P[A \cup B \cup C]$.

$$
\begin{aligned}
Pr[A \cup B \cup C] &= Pr[A] + Pr[B] + Pr[C] \\
&= \quad - Pr[A \cap B] - Pr[A \cap C] - Pr[B \cap C] + Pr[A \cap B \cap C] \\
&= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{16} - 0 - 0 + 0 \\
&= \frac{11}{16}
\end{aligned}
$$

The third line follows from the fact that $Pr[A \cap B] = Pr[A]Pr[B]$ for independent A and B, and the fact that C is disjoint from A and B.

For parts d to h, we consider two events A and B such that P(A) = 0.3 and P(B) = 0.4. Compute P(A|B) in each of the following cases:

(d) [3 pts] A and B are independent

$Pr[A|B] = Pr[A] = 0.3$.

(e) [3 pts] A and B are disjoint

$Pr[A|B] = 0$ by definition of disjoint.

(f) [3 pts] $A \implies B$

$Pr[A \cap B] = Pr[A] \implies Pr[A|B] = \frac{Pr[A]}{Pr[B]} = \frac{3}{4}$.

(g) $P[A \cap B] = 0.1$

$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{0.1}{0.4} = \frac{1}{4}$.

(h) [3 pts] $P(A \cup B) = 0.5$

$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B] \implies Pr[A \cap B] = Pr[A] + Pr[B] - Pr[A \cup B] = 0.2$.

$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{0.2}{0.4} = \frac{1}{2}$.

(i) [4 pts] The number of accidents (per month) at a certain factory has a Poisson distribution. If the probability that there is at least one accident is 1/2, what is the probability that there are exactly two accidents?

If the probability of at least one accident is $1/2$, the probability of none is $1/2$.

For a Poisson distribution, $Pr[X=i] = e^{-\lambda}\frac{\lambda^i}{i!}$.
Thus, $Pr[X=0] = \frac{1}{2} \implies e^{-\lambda} = \frac{1}{2}$
$\implies \lambda = \ln 2$.
And $Pr[X=2] = e^{-\ln 2}\frac{(\ln 2)^2}{2} = \frac{(\ln 2)^2}{4}$.

(j) [4 pts] A pair of dice is rolled until either a 4 is rolled (the numbers on the two dice add up to 4) or a 7 is rolled. What is the expected number of rolls needed?

The number of ways to get 4 is 3, and the number of ways to get 7 is 6. The total number of ways to get either is 9. The probability you get either is $9/36 = 1/4$.

The expected number of times before you get either is the expecation of a geometric distribution with parameter $p = 1/4$. Expecation is $\frac{1}{p} = 4$.

(k) [4 pts] There is a test to determine whether one has boneitis, but the test is not always accurate. For those who do have boneitis, the test has an 4 in 5 chance of coming out positive. For those who don't have boneitis, the test has a 1 in 9 chance of coming out positive. Overall, about 10% of people have boneitis.

Suppose the test comes out positive for That Guy. What is the probability That Guy has boneitis?

$$\begin{aligned} Pr[\text{"pos. test"}] &= Pr[\text{"pos test"}|\text{"boneitis"}]Pr[\text{"}boneitis''] + Pr[\text{"pos. test"}|\text{"boneitis"}](1 - Pr[\text{"boneitis"}]) \\ &= \frac{4}{5}\times\frac{1}{10} + \frac{1}{9}\times\frac{9}{10} \\ &= \frac{9}{50} \end{aligned}$$

$$Pr[\text{"boneitis"}|\text{"pos. test"}] = \frac{Pr[\text{"pos. test"}|\text{"boneitis"}]Pr[\text{"boneitis"}]}{Pr[\text{"}pos.test'']} = \frac{\frac{4}{50}}{\frac{9}{50}} = \frac{4}{9}.$$

(l) [4 pts] A hand of 13 cards are chosen (without replacement) at random from a standard deck of 52 poker cards. What is the expected number of four-of-a-kinds that we see from these 13 cards? (No need to evaluate the expression to get a number.)

(Four-of-a-kind is four cards of the same rank. For example, the hand
"A♠A♠A♢A♣K♠K♡K♢K♣2♠2♠2♢2♣6♢"
contains three four-of-a-kinds, namely the aces, the kings and the twos.)

There are two methods. Define 13 indicator random variables X_i whether the 4 cards of the ith rank (a rank is 2 or 3 or ace, etc.) is included in the hand.

The probability that all the cards in a certain rank are included in a hand is

$$\frac{\binom{48}{9}}{\binom{52}{13}}.$$

By linearity of expecation we get the expected number of 4-of-a kinds is

$$13 \times \frac{\binom{48}{9}}{\binom{52}{13}}.$$

An alternate way to do this is to define a random variable for each of $\binom{13}{4}$ subsets of 4 cards from a 13 card hand and compute the probability that those 4 cards have of a single rank $\frac{13}{\binom{52}{4}}$. This gives

$$\binom{13}{4}\frac{13}{\binom{52}{4}}.$$

I would hope they give the same number.

3 3-SAT. [15 pts]

A 3-conjuctive normal form (CNF) formula is a boolean formula consisting of the "and" of a sequence of clauses where each clause consists of the "or" of three literals. For example, $\phi = (x_1 \vee x_2 \vee \overline{x_5}) \wedge (x_5 \vee \overline{x_2} \vee \overline{x_1})$. (No variable can appear twice in a single clause.)

One wishes to find an assignment to the variables to maximize the number of true clauses. The literals work in the natural manner: $x_1 = T$ if and only if $\overline{x_1} = F$. In the example above, the assignment, $x_1 = T, x_2 = T$ and $x_5 = F$ satisfies one clause in ϕ, where $x_1 = T, x_2 = F, x_5 = F$ satisfies two clauses in ϕ.

(a) For a particular formula with n clauses, consider choosing a random assignment to the variables, i.e., $x_i = T$ or $x_i = F$ with equal probability. What is the expected number of satisfied clauses?

All three literals must be assigned false for a clause to *not* be satisfied. Thus, it is false with probability $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

The probability it is satisfied is thus $\frac{7}{8}$.

By linearity of expecation the total number of satisfied clauses is $\frac{7}{8}n$.

(b) Let U be a random variable corresponding to the number of unsatisfied clauses. What is $E(U)$?

The number of unsatisfied clauses, U, is $n - X$ where X is the number of satisfied clauses. Thus, by linearity of expecation the number of unsatisfied clauses is $n - \frac{7}{8}n = \frac{1}{8}n$.

One could do this from first principles (i.e., a clause is not satisfied with probability $\frac{1}{8}$ and thus the expected number is $\frac{1}{8}n$.)

(c) Upper bound the probability that U is larger than $(1+\varepsilon)E(U)$ for $\varepsilon \geq 0$ as a function of ε. (You should give a nontrivial bound here.)

The variable U is a positive random variable so we can use Markov's inequality as follows

$$Pr[X \geq (1+\varepsilon)E(X)] \leq \frac{E(X)}{(1+\varepsilon)E(X)} = \frac{1}{1+\varepsilon}.$$

(d) Consider repeating this experiment until one finds an assignment that leaves at most $(1+\varepsilon)E(U)$ unsatisfied clauses. Give an upper bound on the expected number of repetitions.

The algorithm terminates when *at most* $(1+\varepsilon)E(U)$ are unsatisfied. This happens with probability at least $1 - \frac{1}{1+\varepsilon} = \frac{\varepsilon}{1+\varepsilon}$.

Since each experiment is independent, it follows a geometric distribution with some parameter $p \geq \frac{\varepsilon}{1+\varepsilon}$.

Thus the expected number of repetitions is at most $\frac{1}{p} \leq \frac{1+\varepsilon}{\varepsilon}$.

4 The evolution of a social network. [18 pts]

(We give a simplified analysis of the connectivity of a social network.)

Say one person in a class of n people knows a secret, perhaps where the midterm is. Occasionally a randomly chosen person A **who doesn't know the secret** calls a randomly chosen person B ($B \neq A$) and learns the secret if B knows it.

Let X_2 be a random variable that represents the number of calls (no two calls are simultaneous) until two people know the secret.

(a) What is the distribution of X_2?

The probability that a call is made to the person who knows the secret is $\frac{1}{n-1}$ as there is 1 good recipient out of $n-1$ possibilities.

Thus, X_2, follows the geometric distribution with parameter $p = \frac{1}{n-1}$.

Or,

$Pr[X_2 = i] = \left(1 - \frac{1}{n-1}\right)^{i-1} \left(\frac{1}{n-1}\right)$.

since for X_2 to be i, the first $i-1$ calls must fail to be to the person who knows the secret and the ith must be to the person who knows the secret.

(b) What is $E[X_2]$?

The expectation of a geometrically distributed random variable with parameter p is $\frac{1}{p}$. Thus, $E[X_2] = n-1$

(c) Let X_i be the number of calls needed to go from $i-1$ people knowing the secret to i people. What is $E[X_i]$?

The probability that a call is made to one of the $i-1$ people who know the secret is $\frac{i-1}{n-1}$ as there is $i-1$ good recipient out of $n-1$ possibilities.

Thus, X_i, follows the geometric distribution with parameter $p = \frac{i-1}{n-1}$.

Or,

$$Pr[X_i = k] = \left(1 - \frac{i-1}{n-1}\right)^{k-1} \left(\frac{1}{n-1}\right).$$

since for X_i to be j, the first $k-1$ calls must fail to be to the person who knows the secret and the kth must be to the person who knows the secret.

(d) What is the expected time for everyone to know the secret?

$E[X_i] = \frac{n-1}{i-1}$ as each X_i follows a geometric distribution.

The time for everyone to know is $X_2 + \cdot + X_n$. By linearity of expectation we get that the total expected time is

$$\sum_{i\geq 2}^{n} \frac{n-1}{i-1} = (n-1)\sum_{i\geq 1}^{n-1} \frac{1}{i}.$$

(e) Bound your expression to within a constant factor for large n. Your expression should not have a summation. (You may use $\Theta(\cdot)$ notation, recall that $2n^2-5n+2=\Theta(n^2)$.)

$\sum_{i=1}^{n-1} \frac{1}{i} \approx (\ln n + \gamma)$

Thus, the expression from the previous problem

$$(n-1)\sum_{i\geq 1}^{n-1} \frac{1}{i} \text{ is } \Theta(n \ln n).$$